Extension du domaine de la lutte

"Sur la gerbe de Kaletha, la Courbe et l'ensemble de Kottwitz"

used for transfert factors non quasi-split case

Purpose:

* Understand the work of Kaletha on "rigid inner forms"
* Extend the geometrization Conjecture obtained w/ Scholze to reach all inner forms of a given reductive group

$\rightsquigarrow$ Geometrization Conjecture for any gp, for example $\underbrace{(D^\times)}_{}^1$, $D/\mathbb{Q}_p$ div. algebra

not an extended pure inner form of SL_n

$\overline{\mathbb{Q}}_p | E | \mathbb{Q}_p$.

$$u = \varprojlim_{\substack{E'|E \\ n \geq 1}} \mathrm{Res}_{E'|E}\, \mu_n$$

Kaletha: $H^2(E,u) \overset{\text{c.f.t.}}{=} \hat{\mathbb{Z}} \ni \underset{\underset{[\mathrm{Kal}]}{\|}}{-1} \rightsquigarrow$ Gerbe

$$\begin{tikzcd} \mathrm{Kal} \arrow[d, "B u"] \\ \mathrm{Spec}\, E \end{tikzcd}$$

$H^1(E,u) = 0$

$\downarrow$

"rigid"

("no automorphisms": if $F : \mathrm{Kal} \xrightarrow{\sim} \mathrm{Kal}$ 1-iso. then $\exists\, \mathrm{Id} \xrightarrow{\sim} F$ 2-iso. non unique)

$C = \hat{E}$

$$\begin{tikzcd} X_E \ni \infty \arrow[d] \\ \mathrm{Spec}\, E \end{tikzcd} \quad , \quad k(\infty) = C$$

Curve attached to $C^\flat$ over $\mathrm{Spec}\, E$

For $E'|E$ (inside $\overline{\mathbb{Q}}_p$)

$$\begin{tikzcd} X_{E'} \arrow[d] & \ni & \infty_{E'} \arrow[d, mapsto] \\ X_E & \ni & \infty_E \end{tikzcd}$$

and $N_{E'|E}\left(\mathcal{O}_{X_{E'}}(\infty_{E'})\right) = \mathcal{O}_{X_E}(\infty_E)$

$\Rightarrow$ if $t = \varprojlim_{E'|E} \mathrm{Res}_{E'|E}\, \mathbb{G}_m$, $1 \to u \to \tilde{E} \to t \to 1$

$\varprojlim_{n \geq 1} t$ = universal cover

$\| $

"$\pi_1(H)$"

②

$\left(\mathcal{O}_{X_{E'}}(\infty_{E'})\right)_{E'/E}$ defines a <u>Canonical</u>

t-torsor
$$\begin{array}{c} T \\ \downarrow \;) \, t \\ X_E \end{array}$$

gerb of roots of T

Prop. $\mathcal{X} = [T/\tilde{t}]$ is isomorphic to $X_E \times_{\operatorname{Spec} E} \mathrm{Kal}$
$$\begin{array}{c} \mathcal{X} \\ \downarrow \;) \, B\mu \\ X_E \end{array}$$

$\to$ use
$$\begin{array}{ccc} \mathfrak{h}_X & = & \mathfrak{h}_{\mathrm{cft}} \\ \cap & & \cap \\ H^2_{\mathrm{et}}(X, \mu_n) & = & H^2(E, \mu_n) \end{array}$$

Cor. Via $H^2(E, u) = H^2_{\mathrm{proet}}(X, u) \to$ definition of Kal that does not use cft.

<u>The extended Kottwitz set</u>

G/E reductive gp.

$$
\begin{array}{l}
\mathrm{Kott} \\
\downarrow \;) \, BD \qquad \text{gerbe of fiber functors on Isocrystals}/E \\
\mathrm{Spec}\, E
\end{array}
$$

$$B(G) = H^1_{\text{ét}}(\mathrm{Kott}, G) = G\text{-isocrystals}/\sim$$
$$= G(\breve{E}) / b \sim g b g^{-\sigma}$$

If $G \overset{u}{\twoheadrightarrow} G'$ with $\ker u$ central then

$\Rightarrow$ $$B(G) \longrightarrow B(G') \xrightarrow{\partial} H^2_{\text{ét}}(\mathrm{Kott}, \ker u)$$

But ~~if ker u a torus~~ $H^2_{\text{ét}}(\mathrm{Kott}, T) = 0$ for T a torus

$\Rightarrow B(G) \twoheadrightarrow B(G')$ surjective if $\ker u$ is connected

False for a non connected diagonalisable group: $H^2_{\text{ét}}(\mathrm{Kott}, D) \neq 0$ in general.

$\Rightarrow$ If Z_G connected then $B(G)_{bas} \twoheadrightarrow B(G_{ad})_{bas} = H^1(E, G_{ad})$

$\rightarrow$ any inner form is an extended pure inner form.

False for any G, for example SL_n.

Recall: X topos, A abelian gp. in X, $\begin{array}{l}\mathcal{X} \\ \downarrow \big) BA \text{ gerbe in } X \\ e \nwarrow \text{final object in } X\end{array}$

H a group in X

$$1 \to H^1(X,H) \underset{\text{pullbacks}}{\longrightarrow} H^1(\mathcal{X},H) \xrightarrow{\partial} H^0\big(X, \mathcal{H}om(A,H)/\underset{\text{H-conjugacy}}{H}\big)$$

(Low degree Leray spectral sequence)

Ex: $1 \to \underbrace{H^1(E,G)}_{\substack{\text{unit root} \\ G\text{-isocrystals}}} \to B(G) \to \big[\mathcal{H}om(\mathbb{D}_E, G_E)/G_E\big]^{\Gamma}$

$[b] \longmapsto [\nu_b]$

Def: $B_e(G) = \{c \in H^1_{et}(\underbrace{\mathrm{Kott} \times \mathrm{Kal}}_{\text{gerbe banded by } \mathbb{D}\times u}, G) \,/\, \lambda_c : u \to Z_G\}$

$\swarrow$

$\{G\text{-}\underbrace{\text{extended isocrystals}}_{\text{Tannakian category of v.b. on Kott} \times \text{Kal}}\}/\sim$

Prop: $1 \to B(G) \to B_e(G) \xrightarrow{\lambda} \mathcal{H}om(u, Z_G) \to 1$

Exact sequence of pointed sets.

Ex: $1 \to B(SL_n) \to B_e(SL_n) \to \frac{1}{n}\mathbb{Z}/\mathbb{Z} \to 1$

↳ Can consider $\sqrt[n]{\mathbb{Q}_p(1)}$ in the category of extended isocrystals.

For example

Prop. $B_e(T) = X_*(T)_{\mathbb{Q}} / I_{\Gamma} \cdot X_*(T)$ functorially in a torus T

$$
\begin{array}{ccccccccc}
0 \to & B(T) & \to & B_e(T) & \xrightarrow{\lambda} & \mathrm{Hom}(u, T) & \to 0 \\
& \| & & \| & & \| & \\
0 \to & X_*(T)_{\Gamma} & \to & X_*(T)_{\mathbb{Q}} / I_{\Gamma} \cdot X_*(T) & \to & X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} & \to 0
\end{array}
$$

Kottwitz

[Th: $\forall D$ diagonalisable$/E$ $H^2_{et}(\mathrm{Kott} \times \mathrm{Kal}, D) = 0$

Cor: * $G \twoheadrightarrow G'$ with central kernel, $\boxed{B_e(G) \twoheadrightarrow B_e(G')}$

* In particular if G' inner form of G $\exists [b] \in B_e(G)_{bas}$ s.t. $G' \simeq G_b$

* Thus $G^* \simeq G_b$ for some b basic !!

⟶ everything simplifies. All can be reduced to the quasisplit case

$B_e(G) \simeq B_e(G^*)$ forget isocrystals w.r.t. add. structures II

$B(G) \hookrightarrow B_e(G)$

The map κ

T maximal torus, $\Phi \subset X^*(T)$ roots.

$$\langle \Phi \rangle^* = \{ v \in X_*(T)_{\mathbb{Q}} \mid \forall \alpha \in \Phi \quad \langle v, \alpha \rangle \in \mathbb{Z} \}$$

Define $\pi_1(G)^e_\Gamma := \langle \Phi \rangle^* \Big/ \langle \check{\Phi} \rangle + I_\Gamma . X_*(T)$

W acts trivially thus independant of choice of T

$$0 \to \pi_1(G)_\Gamma \longrightarrow \pi_1(G)^e_\Gamma \to \mathrm{Hom}(u, Z_G) \to 0$$

Cartesian square

Th: κ extends to

$$\begin{tikzcd}
1 \arrow[r] & B(G) \arrow[r] \arrow[d, "\kappa"] \arrow[dr, phantom, "\square"] & Be(G) \arrow[r, "\lambda"] \arrow[d, "\kappa"] & \mathrm{Hom}(u, Z_G) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1(G)_\Gamma \arrow[r] & \pi_1(G)^e_\Gamma \arrow[r] & \mathrm{Hom}(u, Z_G) \arrow[r] & 1
\end{tikzcd}$$

And it induces

$$\begin{tikzcd}
Be(G)_{bsc} \arrow[r, "\sim"] \arrow[dr, phantom, "\square"] & \pi_1(G)^e_\Gamma . \\
B(G)_{bsc} \arrow[r, "\sim"] \arrow[u, phantom, "\cup" rotate=90] & \pi_1(G)_\Gamma \arrow[u, phantom, "\cup" rotate=90]
\end{tikzcd}$$

Extended geometrization Conjecture

Bun_G^e = "Stack of G-bundles on Curve $\times$ Kol"

Th: (1) Bun_G^e = Artin v-stack

(2) $|\mathrm{Bun}_G^e| = B_e(G)$

(3) $\kappa : |\mathrm{Bun}_G^e| \longrightarrow \pi_1(G)^e_\Gamma$ locally constant

$$\Rightarrow \mathrm{Bun}_G^e = \coprod_{\alpha \in \pi_1(G)^e_\Gamma} \underbrace{\mathrm{Bun}_G^{e,\alpha}}_{\simeq \mathrm{Bun}_{G_b}^{e_1=0}}$$ if $\alpha = \kappa(b)$ with b basic

(4) The choice of $G_b \simeq G^*$ with b basic defines

$$\mathrm{Bun}_G^e \simeq \mathrm{Bun}_{G^*}^e$$

pro-torus

$$* \quad \hat{G}_e = (\hat{G})_{sc} \times [Z(\hat{G})^\circ]_{sc} \twoheadrightarrow \hat{G}$$

$$\mathrm{LocSys}_{\hat{G}_e} = [Z^1(W_E, \hat{G}) / \hat{G}_e] \to \mathrm{Spec}\, \overline{\mathbb{Z}}_\ell .$$

Conj: G quasi-split. Fix Whittaker datum

$$\mathrm{Coh}_{Nilp}(\mathrm{LocSys}_{\hat{G}_e}) \simeq D_{lis}(\mathrm{Bun}_G^e, \overline{\mathbb{Z}}_\ell).$$